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# Randomized Strategies and Prospect Theory in a Dynamic Context<sup>†</sup>

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When prospect theory (PT) is applied in a dynamic context, the probability weighting component brings new challenges. We study PT agents facing optimal timing decisions and consider the impact of allowing them to follow randomized strategies. In a continuous-time model of gambling and optimal stopping, Ebert and Strack (2015) show that a naive PT investor with access only to pure strategies *never* stops. We show that allowing randomization can significantly alter the predictions of their model, and can result in voluntary cessation of gambling.

**Keywords:** Behavioral economics, prospect theory, probability weighting, randomized strategies

**JEL Classification:** D03, D81, G02.

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The most distinctive feature of prospect theory (PT) (Tversky and Kahneman (1992)) is that cumulative probabilities are transformed such that individuals overweight tail events (Quiggin (1982)). Whilst probability weighting has been well studied in static models, the challenges involved in extending the theory to a dynamic setting are considerable. As such, our understanding of the implications of PT in a dynamic context is still developing. This paper considers the role of *randomization* in the agents' set of potential strategies and re-examines the impact of probability weighting in a dynamic context. Our motivation for considering randomization is that we are able to provide new insights on the recent striking result of Ebert and Strack (2015).

Building on a discrete-time model of casino gambling studied by Barberis (2012), Ebert and Strack (2015) study the dynamic investment and gambling behavior of PT agents who are naive.<sup>1</sup> In the Ebert and Strack (2015) model, wealth (prior to any decision to stop gambling) follows a Brownian motion. Agents choose a stopping time, so that the payoff is the value of the stopped process; agents evaluate payoffs using PT. Ebert and Strack show that for a wide range of PT specifications the agent who can commit his future self to following a given strategy would prefer a stopping rule based on first exit time of Brownian motion from a well-chosen interval to one of stopping immediately. From this fact they are able to make a drastic inference: *the naive PT agent never stops* - instead he gambles *until the bitter end*.

Ebert and Strack (2015) focus on pure strategies. However, it is well known (Wakker (1994), and Theorem 7.4.1 of Wakker (2010))<sup>2</sup> that in a static setting PT agents may

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<sup>1</sup>At each instant the naive agent evaluates prospects in an identical fashion to an agent who can commit to a fixed stopping rule. But if today the naive agent elects to gamble with the intention of following a given strategy in the future, then he has no way of forcing his future self to follow that strategy. Instead, at each future moment the naive agent re-evaluates all possible strategies and takes a new decision about whether to gamble further, or to stop. The strategy of the naive agent is the instantaneous strategy of a family of agents who can commit, one for each possible future state of the world.

<sup>2</sup>See also Carassus and Rásonyi (2015) and He et al (2016). Early versions of this paper and He et al (2016) make the point that agents can benefit from following randomized strategies in the Barberis

benefit from following randomized strategies.<sup>3</sup> Hence, in our dynamic setting we might expect a PT agent who can randomize his strategy to outperform one who cannot. We assume that the agent can base his decision on whether to stop on the outcome of a random event.<sup>4</sup> In particular, although he cannot commit his future self to follow a given strategy, he can commit his current self to follow strategies which depend on the outcome of a contemporaneous random event.

Ebert and Strack (2015) show how to choose an interval  $[a, b]$  containing the initial wealth  $x$  such that stopping at the first exit time from this interval is strictly preferred to stopping immediately. Their result is true under a mild condition described as probability weighting being stronger than loss aversion, which they demonstrate holds for the most popular weighting functions. They then extend their arguments based on intervals to prove that for agents restricted to pure strategies, the optimal strategy is to follow a never-stopping strategy and to gamble until the bitter end. We argue that a naive agent with a randomization device (the ability to generate a continuous random variable), and with the ability to commit his current self to following strategies which depend on the realization of this device, does not necessarily follow a never-stopping strategy. The randomized strategy of sometimes stopping immediately and sometimes stopping on first exit from  $[a, b]$  is typically a better prospect than simply stopping on first exit from the interval. Hence the analysis of Ebert and Strack is no longer sufficient to conclude that the naive agent never stops, once randomization is allowed. We give a realistic and tractable example for which we calculate the optimal prospect of an agent and show that if the initial value

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(2012) model of casino gambling.

<sup>3</sup>Wakker (1994, 2010) considers mixed strategies in the context of rank dependent utilities and a static choice of prospect and relates the benefit of using mixed strategies to the concavity of the probability weighting function.

<sup>4</sup>The ineffectiveness of randomization without the ability to commit to the outcome of a random event is well known, see Section 4.2 of Machina (1989). If an agent re-evaluates his candidate strategies after being made aware of the outcome of a random event then, if it was optimal to follow a randomized strategy before the outcome is known, it is again optimal to do so after the outcome is known, and a new randomization is required. A naive agent without the ability to commit to the random outcome enters an ‘infinite loop of indecision’.

of the Brownian motion is at the reference level, then the optimal prospect includes an atom at the reference level. A naive agent who can randomize may stop at this level.

Thus the drastic conclusion of Ebert and Strack (2015) that naive PT agents never stop is no longer the unique prediction if we allow for a wider class of strategies. Naive PT agents who can follow randomized strategies may voluntarily stop gambling.

# 1 Optimal stopping and prospect theory

## 1.1 Prospect Theory preferences

If  $X$  is a random variable then the PT value of  $X$  is given by (see Kothiyal et al (2011))

$$\mathcal{E}(X) = \int_{\mathbb{R}_+} w_+(\mathbb{P}(v(X) > y)) dy - \int_{\mathbb{R}_-} w_-(\mathbb{P}(v(X) < y)) dy \quad (1)$$

where the value function  $v : (-\infty, \infty) \mapsto (-\infty, \infty)$  is strictly increasing with  $v(0) = 0$ , and  $w_{\pm}$  are a pair of probability weighting functions  $w_{\pm} : [0, 1] \rightarrow [0, 1]$  which are strictly increasing and continuous and satisfy  $w_{\pm}(0) = 0$  and  $w_{\pm}(1) = 1$ .

As is standard in the PT literature, we assume that  $v$  is continuous,  $v(0) = 0$ ,  $v$  is concave and increasing on  $x > 0$  and convex and increasing on  $x < 0$  and  $v$  satisfies the (simple) loss aversion property  $v(x) + v(-x) < 0$  for all  $x$ . We assume that  $w_{\pm}$  is concave on  $[0, Q_{\pm}]$  and convex on  $[Q_{\pm}, 1]$  for some  $Q_{\pm} \in (0, 1)$ , with  $w_{\pm}(1/2) < 1/2$ . This last condition is the prevailing experimental finding and is a combination of likelihood insensitivity and pessimism (see Wakker (2010)). Then  $w_{\pm}$  is of inverse-S shape.

We assume that the agents' wealth from gambling or trading of assets follows a Brownian motion  $B^x = (B_t^x)_{t \geq 0}$  where the superscript  $x$  denotes the fact that the Brownian motion starts at  $x$ .<sup>5</sup> The strategies available to the agent correspond to stopping times  $\tau$ , representing when to sell an asset or stop gambling, and then the stopped value  $X = B_{\tau}^x$  represents the prospect of the agent. The goal of the agent is to find  $\tilde{V}(x) = \sup_{\tau \in \Lambda} \mathcal{E}(B_{\tau}^x)$ .

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<sup>5</sup>As in Ebert and Strack (2015) our results generalize to time-homogeneous diffusions. However, the continuity of the wealth process is a key assumption.

Here  $\Lambda$  is the set of uniformly integrable<sup>6</sup> stopping times. Xu and Zhou (2013) analyze optimal stopping problems of this type, primarily for prospects involving gains only.<sup>7</sup> One of the insights of Xu and Zhou (2013) is that the search over stopping times can be replaced by a search over random variables. They show that if  $V(x) = \sup_{X: \mathbb{E}[X]=x} \mathcal{E}(X)$  then  $V(x) = \tilde{V}(x)$ . Our goal is to find  $V(x) = \tilde{V}(x)$  and the set of optimizers for  $V$  (prospects or random variables) and  $\tilde{V}$  (stopping times). We concentrate on the case of initial wealth equal to the reference level.

## 1.2 Randomization in continuous-time stopping models

Ebert and Strack (2015) show that a strategy of stopping on first-exit from a well-chosen interval is preferred to stopping immediately and conclude that naive agents restricted to pure strategies follow a never-stopping strategy. For completeness we restate their result as Proposition 4 in the Appendix. We prove in Proposition 5 that in the same set-up as Ebert and Strack (2015) the strategy of stopping on first exit from an interval can typically be improved upon by mixing with sometimes stopping immediately. This means that the conclusion of Ebert and Strack (2015) that a naive agent never stops may not extend to naive agents with access to randomized strategies. When allowing randomized strategies it is no longer sufficient to identify a prospect which is preferred to stopping immediately: instead, we must identify the optimal prospect. If this prospect has mass at the current wealth level, and if the naive agent can commit his current self to a strategy which depends on the outcome of a contemporaneous random event, then such an agent may stop. This is the program of the next section: we give a simple, tractable example in which the optimal prospect for an agent with zero initial wealth has a mass at zero, and therefore in which the naive agent may stop.

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<sup>6</sup>Recall that the set of uniformly integrable stopping times is the set of finite stopping times  $\rho$  such that  $(B_{t \wedge \rho}^x)_{t \geq 0}$  is a uniformly integrable family of random variables.

<sup>7</sup>Xu and Zhou (2013) cover many different shapes of  $v$  and  $w_{\pm}$  but concentrate on the one-sided case of gains (or losses) only. They restrict their remarks about the two-sided case to describing a broad approach for these problems, but they do not have any general results or solve any examples.

## 2 An example: a naive agent may stop

### 2.1 Specification and Optimal Prospect

Assume that the value function takes the form

$$v(x) = \begin{cases} \ln(1+x) & x \geq 0 \\ Kx & x < 0 \end{cases} \quad (2)$$

where  $K > 1$ . For  $w_-$  assume only that it is inverse-S shaped, and let  $q_- = \operatorname{arginf}_{p \in [0,1]} \frac{w_-(p)}{p}$ .

Assume that  $w_+$  is piecewise linear:<sup>8</sup>

$$w_+(p) = \begin{cases} \alpha p & 0 \leq p \leq q_+ \\ \alpha q_+ + \beta(p - q_+) & q_+ < p < \frac{\alpha-1}{\alpha-\beta} + q_+ \\ 1 - \alpha(1-p) & \frac{\alpha-1}{\alpha-\beta} + q_+ \leq p \leq 1 \end{cases} \quad (3)$$

where  $\alpha > 1$ ,  $\beta \in (0, 1)$  and  $q_+$  are constants with<sup>9</sup>  $0 < q_+ < \frac{1-\beta}{2(\alpha-\beta)}$ .

A change of variable in (1) yields an equivalent formulation

$$\mathcal{E}(X) = \int_{\mathbb{R}_+} v'(z) w_+(\bar{F}_X(z)) dz - \int_{\mathbb{R}_-} v'(z) w_-(F_X(z)) dz \quad (4)$$

where  $F_X$  is the cumulative distribution function (CDF) of  $X$  and  $\bar{F}_X(x) = 1 - F_X(x)$ .

Our problem is to find (the optimizer for)

$$V = \sup \left\{ \int_0^\infty \frac{1}{1+x} w_+(\bar{F}_X(x)) dx - K \int_{-\infty}^0 w_-(F_X(x)) dx \right\} \quad (5)$$

where the supremum is taken over random variables  $X$  with mean 0.

**Proposition 1** *Suppose  $w_\pm$  are such that  $q_+ + q_- < 1$  and  $\frac{1-\alpha q_+}{1-q_+} < \frac{K w_-(q_-)}{q_-} < \alpha$ . If  $x = 0$  then the optimal prospect  $\mathcal{P}^* = \mathcal{P}_0^*$  is a discrete random variable with masses of sizes  $(q_-, 1 - q_- - q_+, q_+)$  at the points  $(-\frac{\nu}{q_-}, 0, \frac{\nu}{q_+})$ , where  $\nu = q_+ \left( \frac{\alpha q_-}{K w_-(q_-)} - 1 \right)$ .*

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<sup>8</sup>Piecewise linear weighting functions of this form were proposed by Webb (2015) as a simple generalization of the probability weighting function used in the NEO-expected utility of Chateauneuf et al (2007). Relative to the NEO-expected utility probability weighting functions, a piecewise linear function has the advantage of being continuous.

<sup>9</sup>A condition  $q_+ \leq \frac{1-\beta}{(\alpha-\beta)}$  is required to ensure that  $\frac{\alpha-1}{\alpha-\beta} + q_+ \leq 1$ . The additional factor of  $\frac{1}{2}$  ensures that  $w_+(\frac{1}{2}) < \frac{1}{2}$ .

It follows from the proposition that the optimal prospect when wealth is at the reference level of zero includes a point mass at the reference level.<sup>10,11</sup>

## 2.2 An optimal stopping rule

The conclusion from the previous section is that there are circumstances in which the optimal prospect includes a mass at zero. Typically (as in the example above) there is uniqueness at the level of optimal prospects. But there are many stopping rules an agent might use to attain a given prospect.<sup>12</sup>

One way to achieve the prospect  $\mathcal{P}_0^*$  is to use a stopping rule in which the stopping time is the first time Brownian motion falls below a well-chosen function of the running maximum. The Azéma-Yor (1979) solution of the Skorokhod embedding problem takes this form.

A second way to achieve the prospect  $\mathcal{P}_0^*$  is to wait until Brownian motion hits  $\hat{a} = -\nu \frac{q_+ + q_-}{q_-}$  or  $\hat{b} = \nu \frac{q_+ + q_-}{q_+}$ , and then to stop the first time thereafter that the Brownian motion is at  $-\frac{\nu}{q_-}$ , 0 or  $\frac{\nu}{q_+}$ .

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<sup>10</sup>The crucial ingredient in our set-up that leads to the conclusion of an atom at zero in the optimal prospect is the fact that the value function has finite slope to the right of the reference level. We have considered several other examples, and again found that the optimal prospect contains a mass at the reference level. We considered an example with value function on gains  $v(x) = \min\{x, 1\}$  and a general pair of probability weighting functions  $w_{\pm}$  and discovered that subject to plausible conditions on  $w_{\pm}$  the optimal prospect includes a mass at the reference level. In this example, the satiation of the utility is not driving the result, but allows us to find an explicit solution for general weighting functions.

<sup>11</sup>For the purposes of the discussion in Section 2.3 concerning the behavior of the naive agent, the significant conclusion is that there is an initial wealth level for which the optimal prospect places mass at that wealth level. We have found examples with infinite slope of the value function at zero for which this is the case and for which the naive agent may stop away from the reference level.

<sup>12</sup>In the probability literature, these are known as solutions of the Skorokhod embedding problem. The only cases where there is a unique embedding (within the class of uniformly integrable embeddings) are those where the optimal prospect is either a point mass at the reference level (when stopping immediately is the only strategy) or a pair of point masses at locations  $\check{a} < 0 < \check{b}$  (when the only strategy is to stop on first exit from the interval  $(\check{a}, \check{b})$ ). In general, for dynamic optimal stopping problems under probability weighting we must expect non-uniqueness of the optimal stopping rule.



For an agent with access to a randomization device (in the form of an independent random variable) other stopping rules attain optimality. The simplest is to stop immediately with probability  $1 - q_+ - q_-$  and to otherwise stop the first time the Brownian motion reaches  $-\frac{\nu}{q_-}$  or  $\frac{\nu}{q_+}$ .

A further way to achieve the prospect  $\mathcal{P}_0^*$  is to stop the first time the Brownian motion hits  $-\frac{\nu}{q_-}$  or  $\frac{\nu}{q_+}$ , and also to stop at zero at a constant rate<sup>13</sup>  $\Phi$ , where  $\Phi = \frac{1-q_+-q_-}{\nu}$ . This stopping rule is time-homogeneous and Markovian in the sense that the decision to stop only depends on the current value of the wealth process. In fact it is the unique stopping rule with these properties.

## 2.3 Naive agents in our example

Proposition 1 describes the optimal prospect for the PT agent, and the discussion in Section 2.2 describes some optimal strategies for an agent with zero initial wealth who can commit to a strategy or stopping rule. What then is the strategy followed by the naive agent? The strategy of the naive agent<sup>14</sup> is the instantaneous, time 0, element of the planned strategy of the agent who can commit to a stopping rule.

When the Brownian motion is at the reference level there is a family of optimal strategies for the agent who can commit, and no unique prediction for the behavior of the naive agent. Any behavior at the reference level (including stopping immediately, always continuing, or stopping at a rate) is consistent with the behavior of a naive agent.<sup>15</sup>

Can we make a more precise prediction for the behavior of a naive agent by restricting attention to reasonable strategies? What might be considered a characteristic of a reasonable strategy? One characteristic would be homogeneity in time, so that the same rule is used to stop or otherwise each time the process returns to a given level. A second

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<sup>13</sup>Then, in the absence of stopping at  $-\frac{\nu}{q_-}$  or  $\frac{\nu}{q_+}$ , the probability the process has not been stopped by time  $t$  is  $e^{-\Phi L_t^0}$  where  $(L_u^0)_{u \geq 0}$  is the local time of Brownian motion at zero.

<sup>14</sup>We assume the naive agent has a randomization device and can use randomized strategies which depend on the outcomes of contemporaneous random events.

<sup>15</sup>If only pure strategies are deemed admissible, then the Ebert and Strack (2015) result implies that instantaneous stopping is never optimal.

characteristic would be that the decision to stop, now and in the future, depends on the prevailing level of the wealth process and not on any other factors. This suggests that a naive agent should adopt a Markovian strategy. Within the class of time-homogeneous, Markovian stopping rules there is a unique strategy which attains a given prospect.<sup>16</sup> If we insist that the naive agent chooses strategies from this class then there is a unique prediction for his behavior. Except in trivial cases, the use of a Markovian strategy requires randomization.

Earlier we gave four stopping rules an agent with the ability to commit might use to attain the optimal prospect. The first strategy based on the Azéma-Yor stopping time is Markov in the pair  $(B_t, S_t = \sup_{s \leq t} B_s)_{t \geq 0}$  but it is not Markov in the wealth process  $B$  alone. The second strategy is not time-homogeneous. Although neither Markovian nor time-homogeneous both these strategies are pure, and never involve stopping immediately. A naive agent basing his strategy on these rules never stops, as predicted by Ebert and Strack (2015).

The third proposed strategy for the agent is to stop immediately with probability  $1 - q_+ - q_-$ , and otherwise to stop the first time the wealth process leaves the interval  $(-\frac{\nu}{q_-}, \frac{\nu}{q_+})$ . This requires a randomization device at  $t = 0$ . The strategy is time inhomogeneous since the agent plans to use a different strategy on any future returns to zero. A naive agent following this strategy has a positive probability of stopping *every* time the Brownian motion is at the reference level. Since Brownian motion started at zero returns to zero infinitely often in any positive time interval, the naive agent using this strategy stops the first time the Brownian motion hits zero with probability one. For wealth pro-

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<sup>16</sup>The key result is the following:

**Proposition 2 (Bertoin and Le Jan (1992))** *Let  $F_X$  denote the CDF of a centred random variable  $X$ . Define  $U_X(x) = \int |y - x| F_X(dy)$  and set  $\chi(dx) = \frac{F_X(dx)}{U_X(x)}$  and  $\Gamma_t = \int_{\mathbb{R}} L_t^x \chi(dx)$ , where  $(L_t^x)_{x \in \mathbb{R}, t \geq 0}$  is the local time of a Brownian motion  $B$ . Then if  $T$  is a rate 1, exponential random variable which is independent of  $B$ , and if  $\rho = \inf\{u : \Gamma_u > T\}$  then  $\rho$  is a uniformly integrable stopping time and  $B_\rho$  has the same law as  $X$ .*

cesses started at the reference level, stopping is immediate with probability one, the exact converse to the Ebert and Strack (2015) result of always waiting to the bitter end.<sup>17</sup>

The fourth strategy also requires randomization, but is Markovian and time-homogeneous. The naive agent following this stopping rule stops at the origin at rate  $\Phi$ . The decision about whether to stop or continue depends only on the current value of wealth, and does not depend on any other factors (including time). The discussion of this Section can be summarized as the following result.

**Theorem 3** *Suppose the naive agent uses a Markovian time-homogeneous strategy. Then, for the example in Section 2.1, (more generally, for a non-empty, open set of parameterized examples of the form given in Section 2.1) the agent may stop at the reference level.*

### 3 Conclusion

This paper derives new results on prospect theory in a dynamic context. Our contribution is to revisit the continuous-time model of Ebert and Strack (2015) under an assumption that agents have access to a richer information structure. Ebert and Strack argue that for any reasonable specification of PT preferences, a naive agent prefers to stop on first exit from some interval to stopping immediately. In a general setting there is a mixed strategy which is preferred to the strategy proposed by Ebert and Strack. This strategy may involve stopping at the reference level. Moreover, in an example, we show that the optimal prospect for an agent includes a mass at zero.<sup>18</sup> If the naive agent is able to follow randomized strategies, and is able to commit to the outcome of contemporaneous random events, then the agent may realize this optimal prospect with a strategy which

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<sup>17</sup>The third proposed strategy yields the earliest stopping. Conversely, any pure, never-stopping strategy is the latest stopping rule which is consistent with the behavior of the naive agent.

<sup>18</sup>More generally, our results indicate that a necessary condition for the optimal prospect for an agent with initial wealth equal to the reference level to include an atom at the reference level is that the value function exhibits finite marginal loss aversion.

involves sometimes stopping.<sup>19</sup>

Typically, unlike the case with no probability weighting, the optimal prospect of a PT agent places mass at more than two points. It follows that the agent who is able to commit to a strategy can realize this optimal prospect using a variety of different stopping rules. The probability weighting element of prospect theory brings a new challenge to the analysis of optimal stopping in a dynamic setting: there is a multiplicity of optimal strategies. This applies to both the agent who can commit, and to the naive agent,<sup>20</sup> and may impact on the usefulness of PT as a predictive model in a dynamic setting.

Ebert and Strack (2015) show that under pure strategies, PT preferences and naïveté lead to the conclusion of never stopping. The authors discuss options to evade their never stopping result including dispensing with the probability weighting element of PT or the notion of naïveté. Our results show that a third possibility is to retain probability weighting and naïveté but to allow the agent to use randomized strategies.

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<sup>19</sup>Our model adds to the literature in which randomization is a deliberate choice. Agranov and Ortoleva (2015) document experimental evidence of deliberate randomization as a source of stochastic choice.

<sup>20</sup>There are optimal strategies for the agent who can commit which do not involve stopping immediately. Hence, never stopping remains an equilibrium strategy for the naive agent.

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# A Randomization in continuous-time models

Assume the reference level is zero, and that the initial value of the Brownian motion is at the reference level.

**Proposition 4 (Ebert and Strack (2015))** *Suppose that (in addition to the standard assumptions listed in Section 1.1) the value function  $v$  satisfies  $0 < v'(0-) = Kv'(0+) < \infty$  for some  $K > 1$ .<sup>21</sup> Suppose for the probability weighting functions  $w_{\pm}$  there exists<sup>22</sup> a  $\hat{p} \in (0, 1/2)$  such that*

$$w_+(\hat{p}) > \frac{K\hat{p}}{1 + (K-1)\hat{p}}, \quad w_-(1-\hat{p}) < \frac{1-\hat{p}}{1 + (K-1)\hat{p}}. \quad (6)$$

*Then there exists  $\hat{\epsilon} > 0$  such that the agent with zero initial wealth prefers gambling until his wealth reaches  $\hat{\epsilon}(1-\hat{p})$  or  $-\hat{\epsilon}\hat{p}$  to stopping immediately.*

From this result Ebert and Strack (2015) infer that a naive agent who can only use pure stopping rules always postpones stopping decisions and hence, never stops.<sup>23</sup>

Now we move on to mixed strategies. We show that in the set-up considered in the main body of Ebert and Strack (2015), the agent who can randomize his strategy prefers such a mixture. Thus if the naive agent can mix over strategies then he prefers sometimes stopping over never stopping.

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<sup>21</sup>Under this assumption the value function exhibits finite marginal loss aversion in the sense that  $v'(0+) < v'(0-) < \infty$ . Köbberling and Wakker (2005) introduce this notion of loss aversion and show that it has the advantage of being scale independent. We have added the adjective marginal to distinguish the concept from simple loss aversion  $v(x) + v(-x) < 0$ .

<sup>22</sup>Ebert and Strack (2015) show that a sufficient condition for there to exist a  $\hat{p}$  such that (6) holds is both  $w'_+(0+) > K$  and  $w'_-(1-) > K$ . This property says that extremely unlikely gains are overweighted and extremely likely losses are underweighted, both by more than the loss aversion parameter, ie. *probability weighting is stronger than loss aversion*.

<sup>23</sup>Ebert and Strack (2015) extend their arguments to show that the never stopping conclusion holds in settings with infinite marginal loss aversion such as the Kahneman and Tversky (1979) power S-shaped value function.

**Proposition 5** *Suppose that, in addition to the assumptions of Proposition 4 we have<sup>24,25</sup>*

$$w'_-((1 - \hat{p})) > w'_+(\hat{p}). \quad (7)$$

*Then there exists  $\hat{\theta} \in (0, 1)$  such that the agent with zero initial wealth prefers a randomized strategy of stopping immediately with probability  $(1 - \hat{\theta})$  and otherwise gambling until his wealth reaches  $\hat{\epsilon}(1 - \hat{p})$  or  $-\hat{\epsilon}\hat{p}$  to the pure strategy of gambling until his wealth reaches  $\hat{\epsilon}(1 - \hat{p})$  or  $-\hat{\epsilon}\hat{p}$ .*

Proposition 5 shows that the first exit strategy of Ebert and Strack (2015) can be improved upon by mixing with sometimes stopping immediately, and hence that the argument in Ebert and Strack is not sufficient to conclude that a naive agent never stops if that agent has access to randomized strategies. Instead, we need to identify the optimal prospect.<sup>26</sup> if this prospect has no mass at the reference level, then a naive agent with the ability to randomize would never stop at the reference level. The main focus of the paper is to show that in general this is *not* the case, by giving an example in which the optimal prospect has a mass at zero.

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<sup>24</sup>Since  $w_{\pm}(\frac{1}{2}) < \frac{1}{2}$  we must have  $\int_0^{\frac{1}{2}} w'_-(1 - p)dp > \int_0^{\frac{1}{2}} w'_+(p)dp$  and for any probability weighting functions there must exist a range of  $\hat{p}$  for which (7) holds. If we take the same parameters and functions as in Table W.1 of Ebert and Strack (2015) then for the Tversky and Kahnemann (1992) and Goldstein and Einhorn (1987) weighting functions (7) holds for all  $\hat{p} < \frac{1}{2}$ .

<sup>25</sup>If, in addition to the restrictions on parameters of  $w_{\pm}$  in Section 2.1 and Proposition 1, we add an assumption that  $\alpha > \frac{K}{1+(K-1)q_+}$  and  $w_-(1 - q) < \frac{1-q}{1+(K-1)q}$  for all  $q \leq q_+$  then (6) holds for all  $\hat{p}$  with  $0 \leq \hat{p} \leq q_+$ . If also  $w'_-(1 - q) > \alpha$  for all  $q \in (0, \bar{q})$  for some  $\bar{q} < q_+$  (such a  $\bar{q}$  exists for the Tversky-Kahneman weighting function, and indeed any continuously differentiable weighting function  $w_-$  with  $w'_-(1-) = \infty$ ) then (7) holds for all  $\hat{p}$  with  $\hat{p} \leq \bar{q}$ . Then we have an example which satisfies the conditions of Proposition 4, Proposition 5 and Proposition 1. For this example, a naive agent with no access to randomization and with zero initial wealth will never stop (Proposition 4, Ebert and Strack (2015)). Such an agent can benefit from randomization (Proposition 5) and his optimal prospect includes an atom at zero (Proposition 1).

<sup>26</sup>Azevedo and Gottlieb (2012, Proposition 2) show that for many examples the problem of finding the optimal prospect is ill-posed, in the sense that there is a sequence of prospects such that the PT value diverges to infinity. But, there are examples which are well-posed, and then our results have greater force.

## A.1 Proofs of Propositions 4 and 5

### Proposition 4

Let  $\epsilon$  be a positive constant which later we will treat as a parameter. Let  $b_\epsilon = \epsilon(1 - \hat{p})$  and  $a_\epsilon = -\epsilon\hat{p}$ . Suppose the agent has zero initial wealth. One strategy open to the agent is to gamble until the first time his wealth reaches  $b_\epsilon$  or  $a_\epsilon$  and then to stop. By construction, the probability that the process hits  $b_\epsilon$  before  $a_\epsilon$  is  $\hat{p}$ . Then the value  $H_{\hat{p}}(\epsilon)$  of this strategy is:

$$H_{\hat{p}}(\epsilon) = w_+(\hat{p})v(b_\epsilon) + w_-(1 - \hat{p})v(a_\epsilon) = w_+(\hat{p})v(\epsilon(1 - \hat{p})) + w_-(1 - \hat{p})v(-\epsilon\hat{p}).$$

Note that  $H_{\hat{p}}(0) = 0$ . Now, writing  $H'_{\hat{p}}$  for the derivative with respect to  $\epsilon$ ,

$$H'_{\hat{p}}(0+) = (1 - \hat{p})w_+(\hat{p})v'(0+) - \hat{p}w_-(1 - \hat{p})v'(0-) > \hat{p}(1 - \hat{p}) \left\{ \frac{K}{1 + (K - 1)\hat{p}} - \frac{K}{1 + (K - 1)\hat{p}} \right\} v'(0+) = 0.$$

Hence there exists  $\hat{\epsilon} > 0$  for which  $H_{\hat{p}}(\hat{\epsilon}) > 0$  and for this  $(\hat{p}, \hat{\epsilon})$  the agent prefers to continue (run until wealth first hits  $b_{\hat{\epsilon}}$  or  $a_{\hat{\epsilon}}$ ) over stopping immediately.

### Proposition 5

Suppose the agent stops gambling immediately with probability  $1 - \theta$  and otherwise gambles until his wealth reaches  $b_{\hat{\epsilon}}$  or  $a_{\hat{\epsilon}}$ . Fixing  $\hat{p}, \hat{\epsilon}$  as in Proposition 4, considering  $\theta$  as a variable and writing  $H_{\hat{p}, \hat{\epsilon}}(\theta)$  as the value of the strategy,

$$H_{\hat{p}, \hat{\epsilon}}(\theta) = w_+(\theta\hat{p})v(b_{\hat{\epsilon}}) + w_-(\theta(1 - \hat{p}))v(a_{\hat{\epsilon}}) = w_+(\theta\hat{p})v(\hat{\epsilon}(1 - \hat{p})) + w_-(\theta(1 - \hat{p}))v(-\hat{\epsilon}\hat{p}).$$

Note that  $H_{\hat{p}, \hat{\epsilon}}(0) = 0$  and  $H_{\hat{p}, \hat{\epsilon}}(1) > 0$  by design. Then

$$\begin{aligned} \frac{\partial}{\partial \theta} H_{\hat{p}, \hat{\epsilon}}(\theta) &= \hat{p}w'_+(\theta\hat{p})v(\hat{\epsilon}(1 - \hat{p})) + (1 - \hat{p})w'_-(\theta(1 - \hat{p}))v(-\hat{\epsilon}\hat{p}) \\ &= \hat{p}(1 - \hat{p}) \left[ w'_+(\theta\hat{p}) \frac{v(\hat{\epsilon}(1 - \hat{p}))}{1 - \hat{p}} + w'_-(\theta(1 - \hat{p})) \frac{v(-\hat{\epsilon}\hat{p})}{\hat{p}} \right]. \end{aligned}$$

Since  $v$  is concave on  $x > 0$ ,  $v(x) + v(-x) < 0$  and  $\hat{p} < 1/2$ , we have

$$\frac{v(-\hat{\epsilon}\hat{p})}{\hat{p}} < -\frac{v(\hat{\epsilon}\hat{p})}{\hat{p}} < -\frac{v(\hat{\epsilon}(1 - \hat{p}))}{1 - \hat{p}}.$$

Recall our hypothesis that  $w'_-((1 - \hat{p})) > w'_+(\hat{p})$ . Then

$$\left. \frac{\partial}{\partial \theta} H_{\hat{p}, \hat{\epsilon}}(\theta) \right|_{\theta=1} < \hat{p} \{ w'_+(\hat{p}) - w'_-(1 - \hat{p}) \} v(\hat{\epsilon}(1 - \hat{p})) < 0,$$

and  $H_{\hat{p}, \hat{\epsilon}}(\theta)$  is maximized at some interior point. The agent who can randomize prefers a mixed strategy of sometimes stopping and sometimes waiting until wealth first leaves the interval  $(a_{\hat{\epsilon}}, b_{\hat{\epsilon}})$  to a pure strategy of waiting until wealth first leaves the same interval.



## B The solution in the example of Section 2

Define

$$D(\mu) = \sup_{\int_0^\infty \bar{F}_X(x)dx = \mu = \int_{-\infty}^0 F_X(x)dx} \left\{ \int_0^\infty \frac{1}{1+x} w_+(\bar{F}_X(x))dx - K \int_{-\infty}^0 w_-(F_X(x))dx \right\}.$$

Then  $V = \sup_\mu D(\mu)$ . The expression for  $D(\mu)$  can be rewritten as

$$D(\mu) = \sup_{(h_+, h_-) \in \mathcal{A}_1^2(\mu)} \left\{ \int_0^\infty \frac{1}{1+x} w_+(h_+(x))dx - K \int_0^\infty w_-(h_-(x))dx \right\}$$

where the set  $\mathcal{A}_1^2(\mu)$  is given by

$$\mathcal{A}_\phi^2(\mu) = \{(h_+, h_-) : [0, \infty)^2 \rightarrow [0, \infty)^2, h_\pm \text{ decreasing and right continuous}, h_\pm(0) \leq 1, \int_0^\infty h_\pm(x)dx = \mu, h_+(0) + h_-(0) \leq \phi\}.$$

Then  $D(\mu) \leq \tilde{D}(\mu)$  where

$$\tilde{D}(\mu) = \sup_{(h_+, h_-) \in \mathcal{A}_2^2(\mu)} \left\{ \int_0^\infty \frac{1}{1+x} w_+(h_+(x))dx - K \int_0^\infty w_-(h_-(x))dx \right\}.$$

In calculating  $D(\mu)$  we require the total mass of the target law  $X$  to be less than or equal to one (with the understanding that  $X$  can be made into a random variable with unit total mass by including an atom at zero). In calculating  $\tilde{D}(\mu)$  we make no such requirement. The advantage of considering  $\tilde{D}(\mu)$  is that the problems over gains and losses decouple:

$$\tilde{D}(\mu) = \sup_{h \in \mathcal{A}_1^1(\mu)} \left\{ \int_0^\infty \frac{1}{1+x} w_+(h(x))dx \right\} - K \inf_{h \in \mathcal{A}_1^1(\mu)} \left\{ \int_0^\infty w_-(h(x))dx \right\}$$

where

$$\mathcal{A}_\phi^1(\mu) = \{h : [0, \infty) \rightarrow [0, \infty), h \text{ decreasing and right continuous}, h(0) \leq \phi, \int_0^\infty h(x)dx = \mu\}.$$

Set

$$\tilde{G}(w; \mu) = \sup_{h \in \mathcal{A}_1^1(\mu)} \int_0^\infty \frac{1}{1+x} w(h(x))dx \quad \tilde{L}(w; \mu) = \inf_{h \in \mathcal{A}_1^1(\mu)} \int_0^\infty w(h(x))dx. \quad (8)$$

Then

$$\tilde{D}(\mu) = \tilde{G}(w_+; \mu) - K \tilde{L}(w_-; \mu).$$

Our plan is to solve for  $\tilde{G}(w_+; \mu)$  and  $\tilde{L}(w_-; \mu)$  (and to find respective optimizers  $h_+^\mu, h_-^\mu$ ) and hence to find the  $\mu$  ( $\hat{\mu}$  say) which maximizes  $\tilde{D}(\mu)$ . There are two possibilities. If  $h_+^{\hat{\mu}}(0) + h_-^{\hat{\mu}}(0) >$

1 then the optimizer for  $\tilde{D} = \sup_{\mu} \tilde{D}(\mu)$  is not feasible for  $D$  (and then the optimizer for  $D$  depends on a complicated interplay between  $h_+(0)$  and  $h_-(0)$ ); alternatively if  $h_+^{\hat{\mu}}(0) + h_-^{\hat{\mu}}(0) \leq 1$  then the optimizer for  $\tilde{D} = \sup_{\mu} \tilde{D}(\mu)$  is feasible for  $D$  and hence is an optimizer for  $D$ . The corresponding optimal target law has  $\mathbb{P}(X > x) = h_+^{\hat{\mu}}(x)$  for  $x > 0$  and  $\mathbb{P}(X < x) = h_-^{\hat{\mu}}(x)$  for  $x > 0$ . If  $h_+^{\hat{\mu}}(0) + h_-^{\hat{\mu}}(0) < 1$  then the optimal target law includes an atom of size  $1 - (h_+^{\hat{\mu}}(0) + h_-^{\hat{\mu}}(0))$  at zero.

*Solution of the problem for gains*

Recall that  $w_+$  is piecewise linear. Let  $W$  be the smallest concave majorant of  $w_+$ . We solve the maximization problem for the concave probability weighting function  $W$ . This gives an upper bound for the problem with function  $w_+$ . We show that this bound is attained.

Set  $\gamma = \frac{1-\alpha q_+}{1-q_+}$ . Then  $\gamma \in (0, 1)$  and

$$W(p) = \begin{cases} \alpha p & 0 \leq p \leq q_+, \\ \alpha q_+ + \gamma(p - q_+) & q_+ < p \leq 1. \end{cases}$$

We want to find  $\tilde{G}(W, \mu) = \sup_{h \in \mathcal{A}_1^1(\mu)} \hat{G}_W(\mu, h)$  where

$$\hat{G}_W(\mu, h) = \int_0^\infty \frac{1}{1+x} (\alpha h(x) I_{\{h(x) \leq q_+\}} + (\alpha q_+ + \gamma(h(x) - q_+)) I_{\{h(x) > q_+\}}) dx.$$

Suppose first that  $\mu \leq \left(\frac{\alpha}{\gamma} - 1\right) q_+$ . Set

$$\mathcal{I}(x) = \max_{g \in [0,1]} \left\{ \frac{1}{1+x} (\alpha g I_{\{g \leq q_+\}} + (\alpha q_+ + \gamma(g - q_+)) I_{\{g > q_+\}}) - \frac{\alpha q_+ g}{\mu + q_+} \right\}. \quad (9)$$

Then, for  $h \in \mathcal{A}_1^1(\mu)$ ,

$$\hat{G}_W(\mu, h) \leq \frac{\alpha q_+ \mu}{\mu + q_+} + \int_0^\infty \mathcal{I}(x) dx \quad (10)$$

From the piecewise linear structure of the objective function the maximum in (9) can only occur at  $g \in \{0, q_+, 1\}$  and

$$\mathcal{I}(x) = \max \left\{ 0, \frac{\alpha q_+}{1+x} - \frac{\alpha q_+^2}{\mu + q_+}, \frac{1}{1+x} - \frac{\alpha q_+}{\mu + q_+} \right\}.$$

Then since

$$\left( \frac{\alpha q_+}{1+x} - \frac{\alpha q_+^2}{\mu + q_+} \right) - \left( \frac{1}{1+x} - \frac{\alpha q_+}{\mu + q_+} \right) = \frac{\alpha q_+}{\mu + q_+} (1 - q_+) - \frac{1 - \alpha q_+}{1+x} \geq \gamma (1 - q_+) - (1 - \alpha q_+) = 0$$

we have

$$\mathcal{I}(x) = \alpha q_+ \left( \frac{1}{1+x} - \frac{q_+}{\mu + q_+} \right) I_{\{x \leq \mu/q_+\}}.$$

Then for any  $h \in \mathcal{A}_1^1(\mu)$

$$\hat{G}_W(\mu, h) \leq \frac{q_+ \alpha \mu}{\mu + q_+} + \int_0^{\mu/q_+} \left[ \frac{\alpha q_+}{(1+x)} - \frac{\alpha q_+^2}{\mu + q_+} \right] dx = \alpha q_+ \ln \left( 1 + \frac{\mu}{q_+} \right). \quad (11)$$

Further, if  $h(x) = q_+ I_{\{x < \mu/q_+\}}$  then  $h \in \mathcal{A}_1^1(\mu)$  and there is equality in (11). Hence  $\tilde{G}(W, \mu) = \alpha q_+ \ln \left( 1 + \frac{\mu}{q_+} \right)$ .

Finally, for the non-negative random variable  $X^* = X^*(\mu)$  defined via  $\bar{F}_{X^*}(x) = q_+ I_{\{x < \mu/q_+\}}$  for  $x \in (0, \infty)$  we find

$$\alpha q_+ \ln \left( 1 + \frac{\mu}{q_+} \right) = \int_0^\infty \frac{1}{1+x} w_+(\bar{F}_{X^*}(x)) dx \leq \tilde{G}(w_+, \mu) \leq \tilde{G}(W, \mu) = \alpha q_+ \ln \left( 1 + \frac{\mu}{q_+} \right)$$

and hence  $\tilde{G}(w_+, \mu) = \alpha q_+ \ln \left( 1 + \frac{\mu}{q_+} \right)$ .

Now suppose  $\mu > \left( \frac{\alpha}{\gamma} - 1 \right) q_+$ . Set

$$\begin{aligned} \mathcal{J}(x) &= \max_{g \in [0,1]} \left\{ \frac{1}{1+x} (\alpha g I_{\{g \leq q_+\}} + (\alpha q_+ + \gamma(g - q_+)) I_{\{g > q_+\}}) - \frac{g}{1+\mu} \right\} \\ &= \max \left\{ 0, \frac{\alpha q_+}{1+x} - \frac{q_+}{1+\mu}, \frac{1}{1+x} - \frac{1}{1+\mu} \right\}. \end{aligned} \quad (12)$$

We find

$$\mathcal{J}(x) = \begin{cases} \frac{1}{1+x} - \frac{1}{1+\mu} & 0 \leq x \leq \gamma(1+\mu) - 1, \\ q_+ \left( \frac{\alpha}{1+x} - \frac{1}{1+\mu} \right) & \gamma(1+\mu) - 1 < x \leq \alpha(1+\mu) - 1, \\ 0 & \gamma(1+\mu) - 1 < x, \end{cases}$$

and the optimizer in (12) is  $g = h^*(x)$  where

$$h^*(x) = \begin{cases} 1 & 0 \leq x \leq \gamma(1+\mu) - 1, \\ q_+ & \gamma(1+\mu) - 1 < x \leq \alpha(1+\mu) - 1, \\ 0 & \alpha(1+\mu) - 1 < x. \end{cases} \quad (13)$$

Then for  $h \in \mathcal{A}_1^1(\mu)$  we have

$$\hat{G}_W(\mu, h) \leq \frac{\mu}{1+\mu} + \int_0^\infty \mathcal{J}(x) dx = \alpha q_+ \ln \left( \frac{\alpha}{\gamma} \right) + \ln \gamma(1+\mu).$$

As before, if we define  $X^*$  via  $\bar{F}_{X^*}(x) = h^*(x)$  then

$$\alpha q_+ \ln \left( \frac{\alpha}{\gamma} \right) + \ln \gamma(1+\mu) = \int_0^\infty \frac{1}{1+x} w_+(\bar{F}_{X^*}(x)) dx \leq \tilde{G}(w_+, \mu) \leq \tilde{G}(W, \mu) \leq \alpha q_+ \ln \left( \frac{\alpha}{\gamma} \right) + \ln \gamma(1+\mu)$$

Hence  $\tilde{G}(w_+, \mu) = \alpha q_+ \ln \left( \frac{\alpha}{\gamma} \right) + \ln \gamma(1+\mu)$ .

In summary,

$$\tilde{G}(w_+, \mu) = \begin{cases} \alpha q_+ \ln \left(1 + \frac{\mu}{q_+}\right) & 0 < \mu \leq q_+ \left(\frac{\alpha}{\gamma} - 1\right) \\ \alpha q_+ \ln \left(\frac{\alpha}{\gamma}\right) + \ln \gamma (1 + \mu) & \mu > q_+ \left(\frac{\alpha}{\gamma} - 1\right) \end{cases}$$

Note that  $\tilde{G}(w_+, \cdot)$  is continuously differentiable (and concave) in its second argument. Let  $\mathcal{G}$  be the inverse to the derivative of  $\tilde{G}$ : then  $\mathcal{G}(y) = (\tilde{G}')^{-1}(y) = (\frac{1}{y} - 1)I_{\{y < \gamma\}} + q_+(\frac{\alpha}{y} - 1)I_{\{y \geq \gamma\}}$ .

*Solution of the problem for losses*

Let  $\eta = \min \left(\frac{w_-(p)}{p}\right)$  and let  $q_- = \operatorname{argmin} \left(\frac{w_-(p)}{p}\right)$ . Then  $w_-(h) \geq \eta h$  with equality at zero and  $q_-$ . For  $h \in \mathcal{A}_1^1(\mu)$  we have

$$\int_0^\infty w_-(h(x))dx \geq \int_0^\infty \eta h(x)dx = \eta \mu.$$

Set  $h^*(x) = q_- I_{(x < \frac{\mu}{q_-})}$ . Then  $h^*$  is admissible and  $\int_0^\infty w_-(h^*(x))dx = \frac{\mu}{q_-} w_-(q_-) = \eta \mu$ . Then  $h^*$  is optimal.

*Solution of the problem for gains and losses*

By definition  $\tilde{D} = \sup_\mu \{\tilde{G}(w_+; \mu) - K\tilde{L}(w_-; \mu)\}$ . We find that the supremum over  $\mu$  is attained at  $\nu$  where, recall  $\nu = \mathcal{G}(\frac{Kw_-(q_-)}{q_-}) = q_+ \left(\frac{\alpha q_-}{Kw_-(q_-)} - 1\right)$ . Our condition that  $\frac{Kw_-(q_-)}{q_-} \in (\gamma, \alpha)$  ensures that  $\nu > 0$ . For  $\mu = \nu$ , the solution of the problem for gains is a mass of size  $q_+$  at  $\frac{\nu}{q_+}$ , and the solution of the problem for losses is a mass of size  $q_-$  at  $-\frac{\nu}{q_-}$ . Moreover, by hypothesis  $q_+ + q_- < 1$ . Then  $\tilde{D} = D$  and the optimizer for the problem is the prospect  $\mathcal{P}_0^*$ .